

THE SPACE OF FRAMED FUNCTIONS IS CONTRACTIBLE

To Stephen Smale on his 80th birthday

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Abstract

According to K. Igusa ([Ig84]) a *generalized Morse function* on M is a smooth function $M \rightarrow \mathbb{R}$ with only Morse and birth-death singularities and a *framed function* on M is a generalized Morse function with an additional structure: a framing of the negative eigenspace at each critical point of f . In ([Ig87]) Igusa proved that the space of framed generalized Morse functions is $(\dim M - 1)$ -connected. J. Lurie gave in [Lu09] an algebraic topological proof that the space of framed functions is contractible. In this paper we give a geometric proof of Igusa-Lurie's theorem using methods of our paper [EM00].

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1 Framed Igusa functions

1.1 Main theorem

This paper is written at a request of D. Kazhdan and V. Hinich who asked us whether we could adjust our proof in [EM00] of K. Igusa's h-principle for generalized Morse functions from [Ig84] to the case of framed generalized Morse functions considered by K. Igusa in his paper [Ig87] and more recently by J. Lurie in [Lu09]. We are very happy to devote this paper to Stephen Smale whose geometric construction in [Sm58] plays the central role in our proof (as well as in the proofs of many other h-principle type results.)

Given an n -dimensional manifold W , a *generalized Morse function*, or as we call it in this paper *Igusa function*, is a function with only Morse (A_1) and birth-death (A_2) type singularities. A *framing* ξ of an Igusa function $\varphi : W \rightarrow \mathbb{R}$ is a trivialization of the negative eigenspace of the Hessian quadratic form at A_1 -points which satisfy certain extra conditions at A_2 -points, see a precise definition below.

If the manifold W is endowed with a foliation \mathcal{F} then we call $\varphi : (W, \mathcal{F}) \rightarrow \mathbb{R}$ a *leafwise Igusa function* if restricted to leaves it has only Morse or birth-

death type singularities. A *framing* ξ of a leafwise Igusa function $\varphi : (W, \mathcal{F}) \rightarrow \mathbb{R}$ is a leafwise framing; see precise definitions below.

The following theorem is the main result of the paper. We use Gromov's notation $\mathcal{O}p A$ for an unspecified open neighborhood of a closed subset $A \subset W$.

1.1.1. (Extension theorem) *Let W be an $(n + k)$ -dimensional manifold with an n -dimensional foliation \mathcal{F} . Let $A \subset W$ be a closed (possibly empty) subset and (φ_A, ξ_A) a framed leafwise Igusa function defined on $\mathcal{O}p A \supset A$. Then there exists a framed leafwise Igusa function (φ, ξ) on the whole W which coincides with (φ_A, ξ_A) on $\mathcal{O}p A$.*

Theorem 1.1.1 is equivalent to the fact that the space of framed Igusa functions is contractible, which is a content of J. Lurie's extension (see Theorem 3.4.7 in [Lu09]) of K. Igusa's result from [Ig87]. The current form of the theorem allows us to avoid discussion of the topology on this space, comp. [Ig87].

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1.2 Framed Igusa functions

Objects associated with a leafwise Igusa function. Let $T\mathcal{F}$ denote the n -dimensional subbundle of TW tangent to the leaves of the foliation \mathcal{F} . Let us fix a Riemannian metric on W . Given a leafwise Igusa function (LIF) φ we associate with it the following objects:

- $V = V(\varphi)$ is the set of all its leafwise critical points, i.e. the set of zeros of the leafwise differential $d_{\mathcal{F}}\varphi : W \rightarrow T^*\mathcal{F}$.
- $\Sigma = \Sigma(\varphi)$ is the set of A_2 -points. Generically, V is a k -dimensional submanifold of W which is transversal to \mathcal{F} at the set $V \setminus \Sigma$ of A_1 -points and has the fold type tangency to \mathcal{F} along a $(k - 1)$ -dimensional submanifold $\Sigma \subset V$ of leafwise A_2 -critical points of φ .
- Vert is the restriction bundle $T\mathcal{F}|_V$.

- $d_{\mathcal{F}}^2\varphi$ is the leafwise quadratic differential of φ . It is invariantly defined at each point $v \in V$. $d_{\mathcal{F}}^2\varphi$ can be viewed as a homomorphism $\text{Vert} \rightarrow \text{Vert}^*$. Using our choice of a Riemannian metric we identify the bundles Vert and Vert^* and view $d_{\mathcal{F}}^2\varphi$ as a self-adjoint operator $\text{Vert} \rightarrow \text{Vert}$. This operator is non-degenerate at the points of $V \setminus \Sigma$, and has a 1-dimensional kernel $\lambda \subset \text{Vert}|_{\Sigma}$. Note that λ is tangent to V , and thus we have $\lambda = \text{Vert} \cap TV|_{\Sigma}$.
- $d_{\mathcal{F}}^3\varphi$ is the invariantly defined third leafwise differential, which is a cubic form on λ . For a leafwise Igusa function φ this cubic form is non-vanishing, and hence the bundle λ is trivial and can be canonically oriented by choosing the direction in which the cubic function $d_{\mathcal{F}}^3\varphi$ increases. We denote by λ^+ the unit vector in λ which defines its orientation.

Decomposition of $V(\varphi)$ and splitting of Vert . The index of the leafwise quadratic differential $d_{\mathcal{F}}^2\varphi(v)$, $v \in V$, may take values $0, 1, \dots, n$ for $v \in V \setminus \Sigma$ and $0, 1, \dots, n-1$ for $v \in \Sigma$. Let

$$V \setminus \Sigma = V^0 \cup \dots \cup V^n \quad \text{and} \quad \Sigma = \Sigma^1 \cup \dots \cup \Sigma^{n-1}$$

be the decompositions of $V \setminus \Sigma$ and Σ according to the index. Note that Σ^i is the intersection of the closures of V^i and V^{i+1} . Then for $v \in V^i$ we have the splitting

$$T_v\mathcal{F} = \text{Vert}(v) = \text{Vert}_+^i(v) \oplus \text{Vert}_-^i(v)$$

where $\text{Vert}_+^i(v)$ and $\text{Vert}_-^i(v)$ are the positive and the negative eigenspaces of $d_{\mathcal{F}}^2\varphi(v)$, and for any $\sigma \in \Sigma^i$ we have the splitting

$$T_{\sigma}\mathcal{F} = \text{Vert}(\sigma) = \text{Ver}(\sigma) \oplus \lambda(\sigma) = \text{Ver}_+^i(\sigma) \oplus \text{Ver}_-^i(\sigma) \oplus \lambda(\sigma)$$

($\text{Ver} \neq \text{Vert}$!), where $\text{Ver}_+^i(v)$ and $\text{Ver}_-^i(v)$ are the positive and the negative eigenspaces of $d_{\mathcal{F}}^2\varphi(\sigma)$. For $\sigma \in \Sigma^i$ and $v \in V^i$ we have

$$\lim_{v \rightarrow \sigma} \text{Vert}_+^i(v) = \text{Ver}_+^i(\sigma) \oplus \lambda(\sigma) \quad \text{and} \quad \lim_{v \rightarrow \sigma} \text{Vert}_-^i(v) = \text{Ver}_-^i(\sigma).$$

For $\sigma \in \Sigma^i$ and $v \in V^{i+1}$ we have

$$\lim_{v \rightarrow \sigma} \text{Vert}_-^{i+1}(v) = \text{Ver}_-^i(\sigma) \oplus \lambda(\sigma) \quad \text{and} \quad \lim_{v \rightarrow \sigma} \text{Vert}_+^{i+1}(v) = \text{Ver}_+^i(\sigma).$$

Framing of a leafwise Igusa function. A framing of a leafwise Igusa function φ is an ordered set $\xi = (\xi^1, \dots, \xi^n)$ of unit vector fields in $\text{Vert}(V)$ such that:

- ξ^i is defined (only) over the union $\Sigma^{i-1} \cup V^i \cup \dots \cup \Sigma^{n-1} \cup V^n$;
- $\xi^i|_{\Sigma^{i-1}} = \lambda^+|_{\Sigma^{i-1}}$;
- $(\xi^1, \dots, \xi^i)|_{V^i}$ is an orthonormal framing for Vert_-^i .

In particular, ξ^n is defined only on $\Sigma^{n-1} \cup V^n$ and ξ^1 is defined only on $V \setminus V^0$. The pair (φ, ξ) is called a *framed leafwise Igusa function* (see Fig.1).

The motivation for adding a framing is discussed in [Ig87].

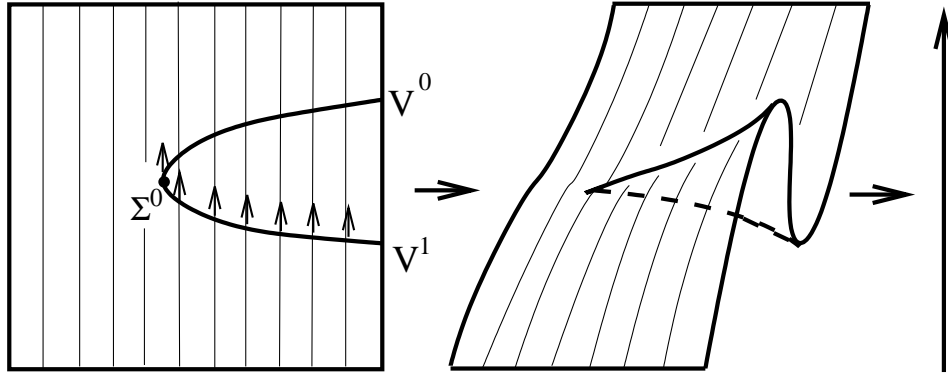


Figure 1: Framed leafwise Igusa function

1.3 Framed formal leafwise Igusa functions

A *formal leafwise Igusa function* (FLIF) is a quadruple $\Phi = (\Phi^0, \Phi^1, \Phi^2, \lambda^+)$ where:

- $\Phi^0 : W \rightarrow \mathbb{R}$ is any function;
- $\Phi^1 : W \rightarrow T\mathcal{F}$ is a vector field tangent to \mathcal{F} , vanishing on a subset $V = V(\Phi) \subset W$;
- Φ^2 is a self-adjoint operator $\text{Vert} \rightarrow \text{Vert}$, which has rank $n - 1$ over a subset $\Sigma = \Sigma(\Phi) \subset V$ and rank n over $V \setminus \Sigma$;
- λ^+ is a unit vector field in the line bundle where $\lambda := \text{Ker}(\Phi^2|_{TV|_{\Sigma}})$.

A leafwise 3-jet of a genuine Igusa function can be viewed as a formal Igusa function Φ , where $\Phi^0 = \varphi$, $\Phi^1 = \nabla_{\mathcal{F}}\varphi$, $\Phi^2 = d_{\mathcal{F}}^2\varphi$ and λ^+ is the unit vector field in $\text{Ker } d_{\mathcal{F}}^2$ oriented by the third differential $d_{\mathcal{F}}^3\varphi$. We denote this FLIF Φ by $J(\varphi)$. A FLIF Φ of the form $J(\varphi)$ is called *holonomic*. Thus we can view a genuine Igusa function as a holonomic formal Igusa function. Usually we will not distinguish between leafwise holonomic functions and corresponding holonomic FLIFs.

Given a FLIF Φ we will use the notation similar to the holonomic case. Namely,

- $V^i \subset V \setminus \Sigma$ is the set of points $v \in V \setminus \Sigma$ where the index (dimension of the negative eigenspace) of Φ_v^2 is equal to i , $i = 0, \dots, n$;
- $\Sigma^i \subset \Sigma$ is the set of points $\sigma \in \Sigma$ such that the index of Φ_σ^2 is equal to i , $i = 0, \dots, n-1$;
- $T_v\mathcal{F} = \text{Vert}(v) = \text{Vert}_+^i(v) \oplus \text{Vert}_-^i(v)$ where $\text{Vert}_+^i(v)$ and $\text{Vert}_-^i(v)$ are the positive and the negative eigenspaces of Φ_v^2 , $v \in V$;
- $T_\sigma\mathcal{F} = \text{Vert}(\sigma) = \text{Ver}(\sigma) \oplus \lambda(\sigma) = \text{Ver}_+^i(\sigma) \oplus \text{Ver}_-^i(\sigma) \oplus \lambda(\sigma)$ where $\text{Ver}_+^i(v)$ and $\text{Ver}_-^i(v)$ are the positive and the negative eigenspaces of Φ_σ^2 , $\sigma \in \Sigma^i$.

As in the holonomic case, for $\sigma \in \Sigma^i$ and $v \in V^i$ we have

$$\lim_{v \rightarrow \sigma} \text{Vert}_+^i(v) = \text{Ver}_+^i(\sigma) \oplus \lambda(\sigma) \quad \text{and} \quad \lim_{v \rightarrow \sigma} \text{Vert}_-^i(v) = \text{Ver}_-^i(\sigma),$$

and so on.

A *framing* for a formal leafwise Igusa function Φ is an ordered set $\xi = (\xi^1, \dots, \xi^n)$ of unit vector fields in $\text{Vert}(V)$ such that:

- ξ^i is defined (only) over the union $\Sigma^{i-1} \cup V^i \cup \dots \cup \Sigma^{n-1} \cup V^n$;
- $\xi^i|_{\Sigma^{i-1}} = \lambda^+|_{\Sigma^{i-1}}$;
- $(\xi^1, \dots, \xi^i)|_{V^i}$ is an orthonormal framing for Vert_-^i .

The pair (Φ, ξ) is called a *framed* formal leafwise Igusa function (framed FLIF).

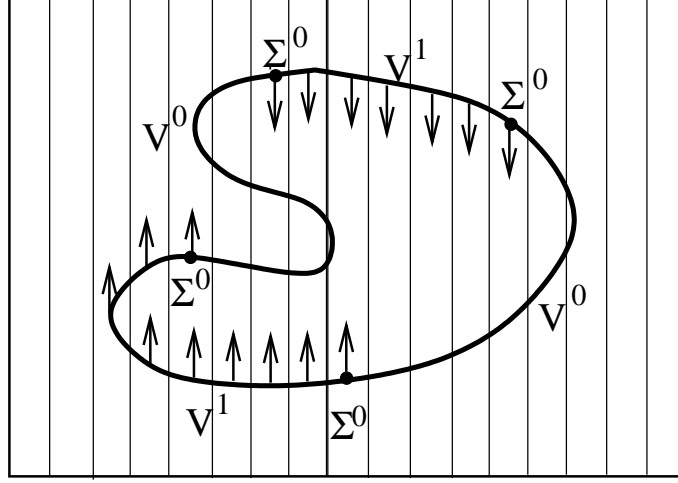


Figure 2: Framed FLIF

As in the holonomic case, for a generic FLIF Φ the set V is a k -dimensional manifold and Σ its codimension 1 submanifold. However, Σ has nothing to do with tangency of V to \mathcal{F} , and moreover there is no control of the type of the tangency singularities between V and \mathcal{F} (see Fig.2).

In what follows we will need to consider FLIFs for different foliations on W . We will say that Φ is an \mathcal{F} -FLIF when we need to emphasize the corresponding foliation \mathcal{F} . Moreover, the notion of a FLIF can be generalized without any changes to an arbitrary, not necessarily integrable n -dimensional distribution $\zeta \subset TW$. We will call such an object a ζ -FLIF. In the case when a distribution ζ is integrable and integrates into a foliation \mathcal{F} we will use as synonyms both terms: ζ -FLIF and \mathcal{F} -FLIF.

Push-forward operation for FLIFs. Let $\zeta, \tilde{\zeta}$ be two n -dimensional distributions in TW . Let $f : W \rightarrow W$ be a diffeomorphism covered by an isomorphism $F : \zeta \rightarrow \tilde{\zeta}$. Let Φ be a ζ -FLIF. Then we define the push-forward $\tilde{\zeta}$ -FLIF $\tilde{\Phi} = (f, F)_* \Phi = (\tilde{\Phi}^0, \tilde{\Phi}^1, \tilde{\Phi}^2, \tilde{\lambda}^+)$ as

- $\tilde{\Phi}^0(f(x)) := \Phi^0(x), x \in W;$
- $\tilde{\Phi}_{f(x)}^1(F(Z)) = \Phi_x^1(Z), x \in W, Z \in \zeta_x;$
- $\tilde{\Phi}_{f(x)}^2(F(Z)) = F(\Phi_x^2(Z)), x \in V, Z \in \text{Vert}_x = \zeta_x;$

$$- \tilde{\lambda}^+(f(x)) = F(\lambda^+(x)), \quad x \in \Sigma.$$

If Φ is framed then the push-forward operator $(f, F)_*$ transforms its framing ξ to a framing $\tilde{\xi}$ of $\tilde{\Phi}$ in a natural way:

$$- \tilde{\xi}^i(f(x)) = F(\xi^i(x)), \quad x \in V.$$

Note that if ζ and $\tilde{\zeta}$ are both integrable, i.e. tangent to foliations \mathcal{F} and $\tilde{\mathcal{F}}$, $F = df$ and Φ is holonomic i.e. $\Phi = J(\varphi)$ then $\tilde{\Phi}$ is also holonomic, $\tilde{\Phi} = J(\tilde{\varphi})$, where $\tilde{\varphi} = \varphi \circ f^{-1}$.

1.4 Outline of the proof and plan of the paper

Any framed leafwise Igusa function can be extended from $\mathcal{O}p A$ to W *formally*, i.e. as a framed FLIF (Φ, ξ) , see Lemma 3.8.1. This is, essentially, an original Igusa's observation from [Ig87]. We then gradually improve (Φ, ξ) to make it holonomic. Note that unlike the holonomic case, the homotopical data associated with Φ^1 and Φ^2 are essentially unrelated. We formulate the necessary so-called *balancing* homotopical condition for a FLIF to be holonomic, see Section 3.4, and show that one can always make a FLIF (Φ, ξ) balanced via a modification, called *stabilization*, see Section 3.6.

Our next task is to arrange that $V(\varphi)$ has fold type tangency with respect to the foliation \mathcal{F} , as it is supposed to be in the holonomic case. FLIFs satisfying this property, together with certain additional coorientation conditions over the fold, are called *prepared*, see Section 3.1. We observe that for a prepared FLIF one can define a stronger necessary homotopical condition for holonomicity. We call prepared FLIFs satisfying this stronger condition *well balanced*, see Section 3.4.

Given any FLIF Φ one can associate with it a *twisted normal bundle* (also called *virtual vertical bundle*) ${}^\Phi\text{Vert}$ over $V = V(\Phi)$ which is a subbundle of $TW|_V$ obtained by twisting the normal bundle of V in W near $\Sigma = \Sigma(\Phi)$, see Section 3.2. In the holonomic case we have ${}^\Phi\text{Vert} = \text{Vert}$, see 3.3.1. A crucial observation is that the manifold V has fold type tangency to any extension ζ of the bundle ${}^\Phi\text{Vert}$ to a neighborhood of V , see 3.2.1. Moreover, if Φ is balanced then there exist a global extension ζ of ${}^\Phi\text{Vert}$ and a bundle isomorphism $F : \text{Vert} \rightarrow {}^\Phi\text{Vert}$ homotopic to the identity $\text{Vert} \rightarrow \text{Vert}$ through injective bundle homomorphisms into $TW|_V$ such that the push-forward framed ζ -FLIF $(\tilde{\Phi}, \tilde{\xi}) = (\text{Id}, F)_*(\Phi, \xi)$ is well balanced, see 3.4.4.

If Φ is holonomic on $\mathcal{O}p A$ then the bundle ζ and the framed FLIF $(\tilde{\Phi}, \tilde{\xi})$ coincide with $T\mathcal{F}$ and (Φ, ξ) over $\mathcal{O}p A$.

The homotopy of the homomorphism F generates a homotopy of distributions ζ_s connecting ζ and $T\mathcal{F}$. If it were possible to construct a fixed on $\mathcal{O}p A$ isotopy V_s of V in W keeping V_s folded with respect to ζ_s then one could cover this homotopy by a fixed on $\mathcal{O}p A$ homotopy of framed well balanced ζ_s -FLIFs $(\tilde{\Phi}_s, \tilde{\xi}_s)$ beginning with $(\tilde{\Phi}_0, \tilde{\xi}_0) = (\tilde{\Phi}, \tilde{\xi})$. Though this is, in general, impossible, the wrinkling embedding theorem from [EM09] allows us to do that after a certain additional modification of V , called *pleating*, see Theorem 2.2.1. We then show that the pleating construction can be extended to the class of framed well balanced FLIFs, see Section 3.5. Thus we get a framed well balanced FLIF $(\hat{\Phi}, \hat{\xi})$ extending the local framed leafwise Igusa function (φ_A, ξ_A) .

The proof now is concluded in two steps. First, we show, see Lemma 3.9.1, that a framed well balanced FLIF can be made holonomic near V , and then use the wrinkling theorem from [EM97] to construct a holonomic extension to the whole manifold W , see Step 5 in Section 4.

The paper has the following organization. In Section 2.1 we discuss the notion of fold tangency of a submanifold with respect to a not necessarily integrable distribution, define the pleating construction for submanifolds and formulate the main technical result, Theorem 2.2.1, which is an analog for folded maps of Gromov's directed embedding theorem, see [Gr86]. This is a corollary of the results of [EM09]. Section 3 is the main part of the paper. We define and study there the notions and properties of balanced, prepared and well balanced FLIFs, and gradually realize the described above program of making a framed FLIF well balanced, see Proposition 3. We also prove here Igusa's result about existence of a formal extension for framed FLIFs, see 3.8.1, and local integrability of well balanced FLIFs, see 3.9.1. Finally, in Section 4 we just recap the main steps of the proof.

2 Tangency of a submanifold to a distribution

In this section we always denote by V an n -dimensional submanifold of an $(n+k)$ -dimensional manifold W , by Σ a codimension 1 submanifold of V and by $\text{Norm} = \text{Norm}(V)$ the normal bundle of V .

2.1 Submanifolds folded with respect to a distribution

Let ζ be an n -dimensional distribution, i.e. a subbundle $\zeta \subset TW$. The *non-transversality* condition of V to ζ defines a variety Σ_ζ of the 1-jet space $J^1(V, W)$. We say that V has at a point $p \in V$ a tangency to ζ of *fold* type if

- $\text{Corank } \pi^\zeta|_{T_p V} = 1$;
- $J^1(j) : V \rightarrow J^1(V, W)$, where $j : V \hookrightarrow W$ is the inclusion, is transverse to Σ_ζ ; we denote $\Sigma := (J^1(j))^{-1}(\Sigma_\zeta)$;
- $\pi^\zeta|_{T_p \Sigma} : T_p \Sigma \rightarrow TW_p/\zeta$ is injective.

If ζ is integrable, and hence locally is tangent to an affine foliation defined by the projection $\pi : \mathbb{R}^{n+k} \rightarrow \mathbb{R}^k$, these conditions are equivalent to the requirement that the restriction $\pi|_V$ has fold type singularity, and in this case one has a normal form for the fold tangency.

If V has fold type tangency to ζ along Σ then we say that V is *folded* with respect to V along Σ . The fold locus $\Sigma \subset V$ is a codimension one submanifold, and at each point $\sigma \in \Sigma$ the 1-dimensional line field $\lambda = \text{Ker } \pi^\zeta|_{TV} = \zeta|_V \cap TV$ is transverse to Σ .

The hyperplane field $T\Sigma \oplus \zeta|_\Sigma$ can be canonically cooriented. In the case when ζ is integrable this coorientation can be defined as follows. The leaves of the foliation through points of Σ form a hypersurface which divides a sufficiently small tubular neighborhood Ω of Σ in W into two parts, $\Omega = \Omega_+ \cup \Omega_-$, where Ω_- is the part which contains $V \cap \Omega$. Then the *characteristic coorientation* of the fold Σ is the coorientation of the hyperplane $T\Sigma \oplus \zeta|_\Sigma$ determined by the outward normal vector field to Ω_- along Σ , see Fig.3. For a general ζ

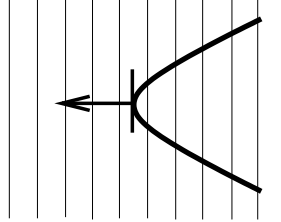


Figure 3: Characteristic coorientation of the fold

take a point $\sigma_0 \in \Sigma$, a neighborhood U of σ_0 in V , an arbitrary unit vector

field $\nu^+ \in (T\Sigma \oplus \zeta|_\Sigma)^\perp$ and consider an embedding $g : U \times (-\epsilon, \epsilon) \rightarrow W$ such that $g(x, 0) = x$, $x \in U$ and $\frac{\partial g}{\partial t}(\sigma, 0) = \nu^+$, $\sigma \in \Sigma \cap U$, where $t \in (-\epsilon, \epsilon)$ is the coordinate corresponding to the second factor. Consider the line field $L = dg(\zeta)$. Note that $L|_{(\Sigma \cap U) \times 0} = \lambda$. The line field L integrates to a 1-dimensional foliation on $U \times (-\epsilon, \epsilon)$ which has a tangency of fold type to $U \times 0$ along $\Sigma \times 0$. Hence, $U \times 0 \subset U \times (-\epsilon, \epsilon)$ can be cooriented, as in the integrable case, which gives the required coorientation of $T\Sigma \oplus \zeta|_\Sigma$, see Fig. 3.

It is important to note that *the property that V has a fold type tangency to ζ along Σ depends only on $\zeta|_V$, and not on its extension to $\mathcal{O}p V$* . Similarly, the above definition of the characteristic coorientation of $T\Sigma \oplus \zeta|_\Sigma$ is independent of all the choices and depends only on $\zeta|_V$ and not on its extension to $\mathcal{O}p V$.

The following simple lemma (which we do not use in the sequel) clarifies the geometric meaning of the fold tangency.

2.1.1. (Local normal form for fold type tangency to a distribution)

Suppose $V \subset W$ is folded with respect to ζ along V and the fold Σ is cooriented. Denote $\lambda := \zeta|_\Sigma \cap TV|_\Sigma$ and $\eta := (\zeta|_V)/\lambda$. Consider the pull-back $\tilde{\eta}$ of the bundle η to $\Sigma \times \mathbb{R}^2$ and denote by E the total space of this bundle. Then there exists a neighborhood Ω of $\Sigma \times 0$ in E , a neighborhood $\Omega' \supset \Sigma$ in W , and a diffeomorphism $\Omega \rightarrow \Omega'$ introducing coordinates (σ, x, z, y) in Ω' , $\sigma \in \Sigma$, $(x, z) \in \mathbb{R}^2$, $y \in \eta$, such that in these coordinates the manifold V is given by the equations $z = x^2, y = 0$ and the bundle $\zeta|_V$ coincides with the restriction to V of the projection $(\sigma, x, z, y) \rightarrow (\sigma, z)$.

Lemma 2.1.1 implies, in particular, that if V is folded with respect to ζ then $\zeta|_V$ always admits an integrable extension to a neighborhood of V .

2.2 Pleating

Suppose V is folded with respect to ζ along Σ . Let $S \subset V \setminus \Sigma$ be a closed codimension 1 submanifold and $\nu^+ \in \zeta$ be a vector field defined over $\mathcal{O}p S \subset W$. For a sufficiently small $\epsilon, \delta > 0$ there exists an embedding $g : S \times [-\delta, \delta] \times [-\epsilon, \epsilon] \rightarrow W$ such that

- $\frac{\partial g}{\partial u}(s, t, u) = \nu^+(g(s, t, u))$, $(s, t, u) \in S \times [-\delta, \delta] \times [-\epsilon, \epsilon]$,
- $g|_{S \times 0 \times 0}$ is the inclusion $S \hookrightarrow V$,

- $g|_{S \times [-\delta, \delta] \times 0}$ is a diffeomorphism onto the tubular δ -neighborhood $U \supset S$ in V , which sends intervals $s \times [-\delta, \delta] \times 0$, $s \in S$, to geodesics normal to S .

Let $\Gamma \subset P := [-1, 1] \times [-1, 1]$ be an embedded connected curve which near ∂P coincides with the line $\{u = 0\}$. Here we denote by t, u the coordinates corresponding to the two factors. We assume that Γ is folded with respect to the foliation defined by the projection $(t, u) \mapsto t$ (this is a generic condition). We denote by $\Gamma_{\delta, \epsilon}$ the image of Γ under the scaling $(t, u) \mapsto (\delta t, \epsilon u)$. Consider a manifold \tilde{V} obtained from V by replacing the neighborhood U by a deformed neighborhood $\tilde{U}_\Gamma = g(S^{n-1} \times \Gamma_{\delta, \epsilon})$. We say \hat{V} is the result of Γ -pleating of V over S in the direction of the vector field ν^+ , see Fig. 4.

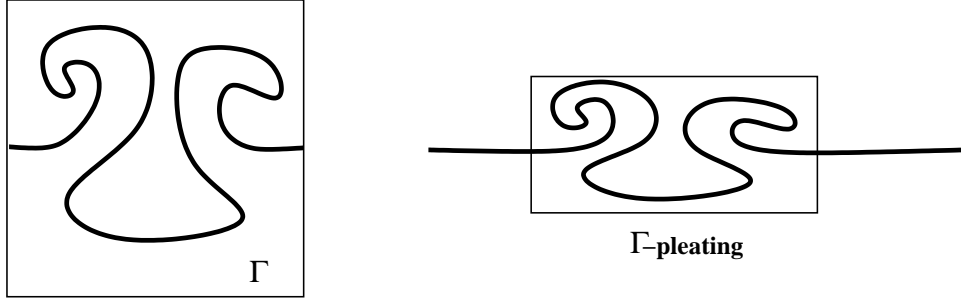


Figure 4: Γ -pleating

The Γ_0^+ -pleating with the curve Γ_0^+ shown on Fig. 5 will be referred simply as *pleating*.

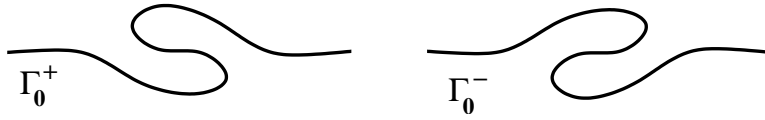


Figure 5: Curves Γ_0^\pm

2.2.1. (Pleated isotopy) Suppose $V \subset (W, \zeta)$ is folded with respect to ζ along $\Sigma \subset V$. Let ζ_s , $s \in [0, 1]$, be a family of n -dimensional distributions over a neighborhood $\Omega \supset V$. Then there exist

- a manifold $\tilde{V} \subset \Omega$ obtained from V by a sequence of pleatings over boundaries of small embedded balls in the direction of vector fields which extend to these balls, and

- a C^0 -small isotopy $h_s : \tilde{V} \rightarrow \Omega$,

such that for each $s \in [0, 1]$ the manifold $h_s(\tilde{V})$ has only fold type tangency to ζ_s . If $\tilde{\Sigma} = \Sigma \cup \Sigma'$ is the fold of \tilde{V} with respect to ζ_0 then $h_s(\tilde{\Sigma})$ is the fold of $h_s(\tilde{V})$ with respect to ζ_s . If the homotopy ζ_s is fixed over a neighborhood $\mathcal{O}p A$ of a closed subset $A \subset V$ then one can arrange that $V \cap \mathcal{O}p A = \tilde{V} \cap \mathcal{O}p A$ and the isotopy h_s is fixed over $\mathcal{O}p A$.

Theorem 2.2.1 is a version of the wrinkled embedding theorem from [EM09], see Theorem 3.2 in [EM09] and the discussion in Sections 3.2 and 3.3 in that paper on how to replace the wrinkles by spherical double folds and how to generalize Theorem 3.2 to the case of not necessarily integrable distributions. Another cosmetic difference between the formulations in [EM09] and Theorem 2.2.1 is that the former one allows not only double folds, but also their embryos, i.e. the moments of death-birth of double folds. This can be remedied by preserving the double folds till the end in the near-embryo state, rather than killing them, and similarly by creating the necessary number of folds by pleating at the necessary places before the deformation begins.

2.2.2. (Remark) If \tilde{V} satisfies the conclusion of Theorem 2.2.1 then any manifold $\tilde{\tilde{V}}$ obtained from \tilde{V} by an additional Γ -pleating with *any* Γ will also have this property. For our purposes we will need to pleat with three special curves Γ_1 and Γ_2^\pm shown on Figure 6. As it clear from this picture, a pleating

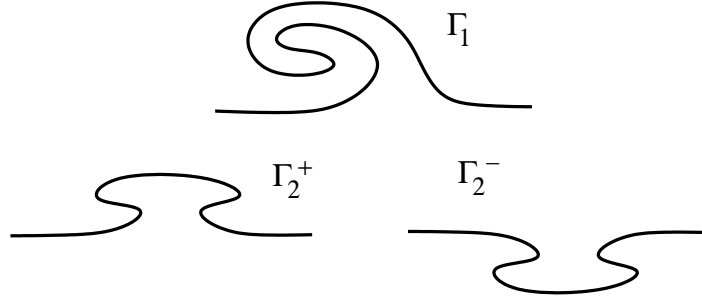


Figure 6: Curves Γ_1 and Γ_2^\pm

with any of these curves can be viewed as a result of a Γ_0^+ -pleating followed by a second Γ_0^- -pleating. Hence, in the formulation of Theorem 2.2.1 one can pleat with any of the curves Γ_1 and Γ_2^\pm instead of Γ_0^+ .

3 Geometry of FLIFs

3.1 Homomorphisms Γ_Φ and Π_Φ

Given a ζ -FLIF Φ we will associate with it several objects and constructions.

Isomorphism $\Gamma_\Phi : \text{Norm} \rightarrow \text{Vert}$. This isomorphism is determined by Φ^1 . The tangent bundle $T(\zeta|_V)$ to the total space of the bundle $\zeta|_V$ canonically splits as $\text{Vert} \oplus TW|_V$, and hence the bundle of tangent planes to the section Φ^1 along its 0-set V can be viewed as a graph of a homomorphism $\widehat{\Gamma}_\Phi : TW|_V \rightarrow \text{Vert}$ vanishing on TV . The restriction of this homomorphism to Norm will be denoted by Γ_Φ . The transversality of the section Φ^1 to the 0-section ensures that $\text{Ker } \widehat{\Gamma}_\Phi = TV$ and hence Γ_Φ is an isomorphism.

By an *index* coorientation of Σ in V we will mean its coorientation by a normal vector field τ^+ pointing in the direction of **decreasing** of the index, i.e. on Σ^i it points into V^i . We will denote by n^+ the vector field $\Gamma_\Phi^{-1}(\lambda^+) \in \text{Norm}(V)$, see Fig. 7.

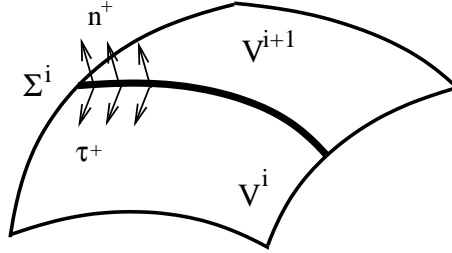


Figure 7: The vector fields n^+ and τ^+

In the holonomic situation the index coorientation is given by the vector field λ^+ and the vector field n^+ determines the characteristic coorientation of the fold.

We call a ζ -FLIF Φ *prepared* if

- $V(\Phi)$ is folded with respect to ζ with the fold along $\Sigma(\Phi)$;
- $TV \cap \text{Vert}_\Sigma = \lambda$ and the vector field λ^+ determines the *index* coorientation of the fold;
- the vector field n^+ determines the *characteristic* coorientation of the fold Σ .

Thus, any *holonomic* FLIF (when, in particular, ζ is integrable) is prepared.

Isomorphism $\Pi_\Phi : \text{Norm} \rightarrow \text{Vert}$. Given a *prepared* ζ -FLIF Φ , let us denote by K the restriction of the orthogonal projection $TW|_V \rightarrow \text{Norm}$ to the subbundle $\text{Vert} = \zeta|_V \subset TW|_V$. The homomorphism K is non-degenerate over $V \setminus \Sigma$ and has a 1-dimensional kernel λ over Σ .

3.1.1. (Definition of Π_Φ) *The composition $\Phi^2 \circ K^{-1} : \text{Norm}|_{V \setminus \Sigma} \rightarrow \text{Vert}|_{V \setminus \Sigma}$ continuously extends to a non-degenerate homomorphism $\Pi_\Phi : \text{Norm} \rightarrow \text{Vert}$.*

Proof. Let us prove the extendability of the inverse operator $K \circ (\Phi^2)^{-1}$. There exists a canonical extension of the vector field λ^+ as a unit Φ^2 -eigenvector field $\tilde{\lambda}^+$ on $\mathcal{O}p \Sigma \subset V$. Then

$$\Phi^2(\tilde{\lambda}^+(v)) = c(v)\tilde{\lambda}^+(v), \quad v \in \mathcal{O}p \Sigma,$$

where the eigenvalue function $c : \mathcal{O}p \Sigma \rightarrow \mathbb{R}$ has Σ as its regular 0-level. Denote $\widetilde{\text{Ver}} = \tilde{\lambda}^\perp(v)$ the orthogonal eigenspace of $\Phi^2(v)$. Denote $\widetilde{\text{Nor}} := K(\widetilde{\text{Ver}})$. The operator $K \circ (\Phi^2)^{-1}$ is well defined on $\text{Ver} = \widetilde{\text{Ver}}|_\Sigma \subset \text{Vert}|_\Sigma$ and maps it isomorphically onto $\text{Nor} = \widetilde{\text{Nor}}|_\Sigma$. It remains to prove existence of a non-zero limit

$$\lim_{v \rightarrow v_0 \in \Sigma} K \left((\Phi^2)^{-1}(\tilde{\lambda}^+) \right) = \lim_{v \rightarrow v_0 \in \Sigma} \frac{1}{c(v)} K(\tilde{\lambda}^+(v)).$$

The vector-valued function $K(\tilde{\lambda}^+(v))$ vanishes on Σ while the function $c(v)$ has no critical points on Σ . Hence, the above limit exists. On the other hand, the transversality condition for the fold implies that $\|K(\tilde{\lambda}^+(v))\| \geq a \text{dist}(v, \Sigma)$, while $|c(v)| \leq b \text{dist}(v, \Sigma)$ for some positive constants $a, b > 0$, and therefore $\lim_{v \rightarrow v_0 \in \Sigma} \frac{1}{c(v)} K(\tilde{\lambda}^+(v)) \neq 0$. \square

3.1.2. *If Φ is holonomic then $\Pi_\Phi = \Gamma_\Phi$.*

Proof. Indeed, recall that $\Gamma_\Phi = \widehat{\Gamma}_\Phi|_{\text{Norm}}$, where $\widehat{\Gamma}_\Phi : TW|_V \rightarrow \text{Vert}$ is the homomorphism defined by the section Φ^1 linearized along its zero-set V . In the holonomic situation one has over $V \setminus \Sigma$ the equality

$$\widehat{\Gamma}_\Phi|_{\text{Vert}} = d^2\varphi = \Phi^2,$$

where $\varphi = \Phi^0$. But $\widehat{\Gamma}_\Phi|_{\text{Vert}}$ and $\widehat{\Gamma}_\Phi|_{\text{Norm}}$ are related by a projection along the kernel of $\widehat{\Gamma}_\Phi$ which is equal to TV . Hence, $\Gamma_\Phi = \Phi^2 \circ K^{-1} = \Pi_\Phi$. By continuity, the equality $\Pi_\Phi = \Gamma_\Phi$ holds everywhere. \square

3.2 Twisted normal bundle and the isomorphism Δ_Φ

Given any ζ -FLIF Φ we define here a *twisted normal bundle*, or as we also call it *virtual vertical bundle* ${}^\Phi\text{Vert} \subset TW_V$ over V . As we will see later (see 3.3.1), in the holonomic case ${}^\Phi\text{Vert}$ coincides with Vert .

Let $U = \Sigma \times [-\epsilon, \epsilon]$ be the tubular neighborhood of Σ in V of radius $\epsilon > 0$. We assume that the splitting is chosen in such a way that the vector field $\frac{\partial}{\partial t}$, where t is the coordinate corresponding to the second factor, defines the index coorientation of Σ in V , and hence coincides with τ^+ . We denote

$$U_+ := \Sigma \times (0, \epsilon], \quad U_- := \Sigma \times [-\epsilon, 0).$$

Denote by $\tilde{\lambda}^+ \in \text{Vert}$ the unit eigenvector field of $\Phi^2|_U$ which extends $\lambda^+ \in \text{Vert}|_\Sigma$. If ϵ is small enough then such extension is uniquely defined. Let $\widetilde{\text{Ver}} := \tilde{\lambda}^\perp$ be the complementary Φ^2 -eigenspace. We have $\widetilde{\text{Ver}}|_\Sigma = \text{Ver}$. Denote $\tilde{n}^+ := \Gamma_\Phi^{-1}(\tilde{\lambda}^+)$, $\widetilde{\text{Nor}} = \Gamma_\Phi^{-1}(\widetilde{\text{Ver}})$, $\tilde{\tau}^+ := \frac{\partial}{\partial t}$. Choose a function $\theta : U \rightarrow [-\frac{\pi}{2}, \frac{\pi}{2}]$ which has Σ as its regular level set $\{\theta = 0\}$, and which is equal to $\pm\frac{\pi}{2}$ near $\Sigma \times (\pm\epsilon)$.

We define the bundle ${}^\Phi\text{Vert}$ in the following way. Over $V \setminus U$ it is equal to Norm . The fiber over a point $v \in U$ is equal to $\text{Span}(\widetilde{\text{Nor}}, \mu(v))$, where the line $\mu(v)$ is generated by the vector

$$\mu^+(v) = \sin \theta(v) \tilde{n}^+(v) + \cos \theta(v) \tilde{\tau}^+(v),$$

see Fig. 8.

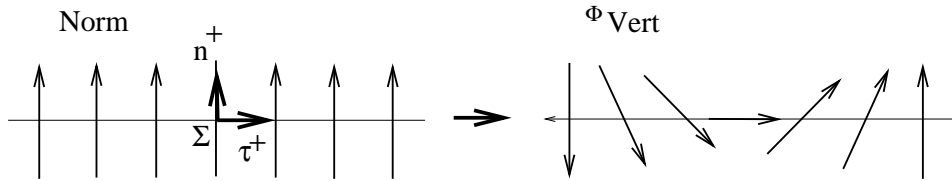


Figure 8: Twisting the normal bundle

Isomorphism $\Delta_\Phi : \text{Vert} \rightarrow {}^\Phi\text{Vert}$. Let $c : U \rightarrow \mathbb{R}$ be the eigenvalue function corresponding to the Φ^2 -eigenvector field $\tilde{\lambda}^+ \in \text{Vert}$ on U , i.e. we have $\Phi^2(\tilde{\lambda}^+(v)) = c(v)\tilde{\lambda}^+(v)$, $v \in U$. The function c is positive on U_+ and negative

on U_- . Let $\tilde{c} : U \rightarrow \mathbb{R}$ be any *positive* function which is equal to c on $\partial U_+ = \Sigma \times \epsilon$ and equal to $-c$ on $\partial U_- = \Sigma \times (-\epsilon)$.

We then define the operator

$$\Delta_\Phi : \text{Vert} \rightarrow {}^\Phi\text{Vert}$$

by the formula

$$\Delta_\Phi(Z) = \begin{cases} \Gamma_\Phi^{-1}(\Phi^2(Z)), & \text{over } V \setminus U, Z \in \text{Vert}; \\ \Gamma_\Phi^{-1}(\Phi^2(Z)), & \text{over } U, Z \in \widetilde{\text{Ver}}; \\ \tilde{c}(v)(\sin \theta(v)\tilde{n}^+ + \cos \theta(v)\tilde{\tau}^+), & Z = \tilde{\lambda}^+(v), v \in U. \end{cases} \quad (1)$$

It will be convenient for us to keep some ambiguity in the definition of ${}^\Phi\text{Vert}$ and Δ_Φ . However, we note that the space of choices we made in the definition is contractible, and hence the objects are defined in a homotopically canonical way.

Let us extend ${}^\Phi\text{Vert}$ and Δ_Φ to a neighborhood $\mathcal{O}p V \subset W$. We will keep the same notation for the extended objects.

3.2.1. *For any ζ -FLIF Φ the ${}^\Phi\text{Vert}$ -FLIF*

$$\Phi^{\text{Norm}} = (\text{Id}, \Delta_\Phi)_* \Phi$$

on $\mathcal{O}p V$ is prepared.

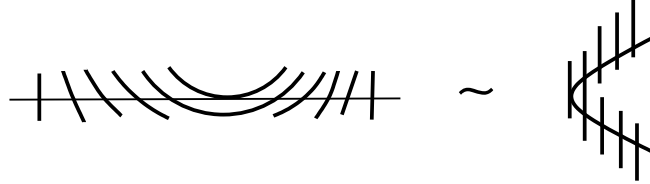


Figure 9: V is folded with respect to ${}^\Phi\text{Vert}$.

Proof. Denote $\widehat{\Phi} := \Phi^{\text{Norm}}$. We have $V(\widehat{\Phi}) = V(\Phi) = V$. First of all we observe (see Fig. 9) that V is folded with respect to ${}^\Phi\text{Vert}$ along Σ and the vector field $n^+ = n^+(\Phi)$ defines the characteristic coorientation of the fold. On the other hand, $\lambda^+(\widehat{\Phi}) = \Delta_\Phi(\lambda_+(\Phi)) = \tau^+(\Phi) = \tau^+(\widehat{\Phi})$ and

$$n^+(\widehat{\Phi}) = \Gamma_{\widehat{\Phi}}^{-1}(\lambda^+(\widehat{\Phi})) = \Gamma_{\widehat{\Phi}}^{-1}(\tau^+) = \Gamma_\Phi^{-1}(\Delta_\Phi^{-1}(\tau_+)) = \Gamma_\Phi^{-1}(\lambda^+) = n^+(\Phi),$$

and hence $n^+(\widehat{\Phi})$ defines the characteristic coorientation of the fold Σ . Thus $\widehat{\Phi}$ is prepared. \square

3.2.2. *For any FLIF Φ the diagram*

$$\begin{array}{ccc} \text{Norm} & \xrightarrow{\Gamma_\Phi} & \text{Vert} \\ & \searrow \Pi_{\widehat{\Phi}} \quad \swarrow \Delta_\Phi & \\ & \Phi\text{Vert} & \end{array}$$

commutes for appropriate choices in the definition of ΦVert and Γ_Φ .

Proof. We need to check that $\Pi_{\widehat{\Phi}} = \Delta_\Phi \circ \Gamma_\Phi$. First, we check the equality over $V \setminus U$. We have $\text{Norm}|_{V \setminus U} = \Phi\text{Vert}|_{V \setminus U}$, and hence $K_{\widehat{\Phi}} = \text{Id}$. Furthermore, over $V \setminus U$ we have $\Pi_{\widehat{\Phi}} = K_{\widehat{\Phi}}^{-1} \circ \widehat{\Phi}^2 = \Delta_\Phi \circ \Phi^2 \circ \Delta_\Phi^{-1} = \Gamma_\Phi^{-1} \circ \Phi^2 \circ \Phi^2 \circ (\Phi^2)^{-1} \Gamma_\Phi = \Gamma_\Phi^{-1} \circ \Phi^2 \circ \Gamma_\Phi = \Delta_\Phi \circ \Gamma_\Phi$. Similarly, we check that $\Pi_{\widehat{\Phi}}|_{\widetilde{\text{Nor}}} = \Delta_\Phi \circ \Gamma_\Phi|_{\widetilde{\text{Nor}}}$. Finally, evaluating both parts of the equality on the vector field n_+ we get: $\Pi_{\widehat{\Phi}}(n^+) = \lambda^+ = \Delta_\Phi(\Gamma_\Phi(n^+))$. Then this implies $\Pi_{\widehat{\Phi}}(\widetilde{n}^+) = \Delta_\Phi(\Gamma_\Phi(\widetilde{n}^+))$ for an appropriate choice of the function $\tilde{c} > 0$ in the definition of the homomorphism Δ_Φ . \square

3.3 The holonomic case

We will need the following normal form for a leaf-wise Igusa function φ near Σ (see [Ar76, El72]).

Consider the pull-back of the bundle $\text{Ver} = \text{Ver}_+ \oplus \text{Ver}_-$ defined over Σ to $\Sigma \times \mathbb{R} \times \mathbb{R}$ via the projection $\Sigma \times \mathbb{R} \times \mathbb{R} \rightarrow \Sigma$. Let E be the total space of this bundle. The submanifold $\Sigma \times 0 \times 0$ of the 0-section of this bundle we will denote simply by Σ . Consider a function $\theta : E \rightarrow \mathbb{R}$ given by the formula

$$\theta(\sigma, x, z, y_+, y_-) = x^3 - 3zx + \frac{1}{2}(\|y_+^2\| - \|y_-^2\|); \quad (2)$$

$(\sigma, x, z) \in \Sigma \times \mathbb{R} \times \mathbb{R}$, $y_\pm \in (\text{Ver}_\pm)_\sigma$.

Consider the projection $p : E \rightarrow \Sigma \times \mathbb{R}$ defined by the formula

$$p(\sigma, x, z, y_+, y_-) = (\sigma, z).$$

There exists an embedding $g : \mathcal{O}p \Sigma \rightarrow W$, where $\mathcal{O}p \Sigma$ is a neighborhood of Σ in E , such that

- $g(\sigma) = \sigma, \sigma \in \Sigma$;
- g maps the fibers of the projection p to the leaves of the foliation \mathcal{F} .
- $\varphi \circ g = \theta$.

Via the parameterization map g we will view (σ, x, z, y_+, y_-) as coordinates in $\mathcal{O}_p \Sigma \subset W$. In these coordinates the function φ has the form (2), the manifold V is given by the equations $z = x^2, y_{\pm} = 0$, the foliation \mathcal{F} is given by the fibers of the projection p , the vector field $-\frac{\partial}{\partial z}$ defines the characteristic coorientation of the fold Σ , and the vector field $\frac{\partial}{\partial x} \in TV|_{\Sigma}$ defines the index coorientation.

The normal form (2) can be extended to a neighborhood of V using the parametric Morse lemma. However, we will not need it for our purposes.

3.3.1. *If Φ is holonomic then for appropriate auxilliary choices the virtual vertical bundle ${}^{\Phi}\text{Vert}$ coincides with Vert and the isomorphism*

$$\Delta_{\Phi} : \text{Vert} \rightarrow {}^{\Phi}\text{Vert} = \text{Vert}$$

is the identity.

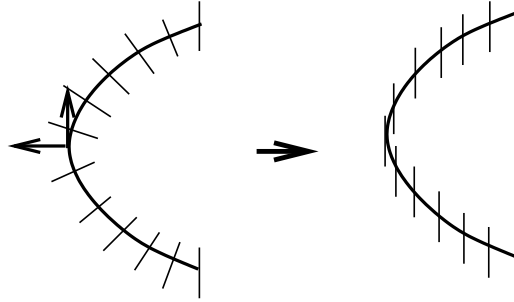


Figure 10: Holonomic case: the bundle ${}^{\Phi}\text{Vert}$ coincides with Vert

Proof. Let Φ be holonomic and $\Phi^0 = \varphi$. The bundle Vert is transverse to V over $V \setminus U$, and over U it splits as $\widetilde{\text{Ver}} \oplus \widetilde{\lambda}$. We have $\widetilde{\text{Nor}} \cap TV = \{0\}$, the bundle $\widetilde{\lambda}$ is tangent to V along Σ and $\lambda = \widetilde{\lambda}|_{\Sigma}$ is transverse to Σ . Let us choose a metric such that the transversality condition for the bundles $\text{Vert}|_{V \setminus U}$, $\widetilde{\text{Ver}}|_U$, $\lambda|_U$ are replaced by the orthogonality one. Then the operator Γ_{Φ}^{-1} , and hence Δ_{Φ} leaves invariant the bundles $\text{Vert}|_{V \setminus U}$ and

$\widetilde{\text{Ver}}|_U$. Moreover, on both these bundles the operators $\Phi^2 = d^2\varphi$ and Γ_Φ coincide, and hence $\Delta_\Phi = \text{Id}$. It remains to analyze $\Delta_\Phi|_{\tilde{\lambda}^+}$. By definition,

$$\Delta_\Phi(\tilde{\lambda}^+(v)) = \tilde{c}(v) (\cos \theta(v) \tau^+(v) + \sin \theta(v) \tilde{n}^+(v)), \quad v \in U,$$

where $\tilde{n}^+ = \Gamma_\Phi^{-1}(\tilde{\lambda}^+)$. It is sufficient to ensure that the line $\Delta_\Phi|_{\tilde{\lambda}}$ coincides with $\tilde{\lambda}^+$ because then the similar equality for vectors could be achieved just by choosing an appropriate amplitude function \tilde{c} in the definition of the operator Δ_Φ . Note that we have $\Delta_\Phi(\tilde{\lambda}(v)) = \lambda(v)$ for $v \in \partial U$ or $v \in \Sigma$. To ensure this equality on the rest of U we need to further specify our choices. As it was explained above in Section 3.3 we can assume that the function φ in a neighborhood $\Omega \supset U$ in W is given by the normal form (2). Choosing $\Omega = \{|x|, |z| \leq \epsilon\}$ we have

$$U := V \cap \Omega = \{z = x^2, y_\pm = 0, |x| \leq \epsilon\},$$

and bundles $\widetilde{\text{Ver}}$, $\widetilde{\text{Ver}}$ and $\tilde{\lambda}$ are given, respectively, by restriction to V of the projections $(\sigma, x, z, y_+, y_-) \mapsto (\sigma, z)$, $(\sigma, x, z, y_+, y_-) \mapsto (\sigma, x, z)$ and $(\sigma, x, z, y_+, y_-) \mapsto (\sigma, z, y_-, y_+)$. Let us choose the tangent to V vector field $\frac{\partial}{\partial x} + 2x\frac{\partial}{\partial z}$ as $\tilde{\tau}^+$ and recall that we have chosen a metric for which the vectors $\tau^+(v)$ and $\tilde{\lambda}^+(v)$ for $v \in \partial U$ are orthogonal. Let us choose any vector field $\hat{v} \in P := \text{Span}(\frac{\partial}{\partial x}, \frac{\partial}{\partial z})$ such that

- $\hat{v}^+|_{\partial U_+} = \tilde{\lambda}^+|_{\partial U_+}$;
- $\hat{v}^+|_{\partial U_-} = -\tilde{\lambda}^+|_{\partial U_-}$;
- $\hat{v}^+|_\Sigma = -\frac{\partial}{\partial z}$ defines the characteristic coorientation;
- the vector field $\lambda^+|_{\text{Int } U^+}$ belongs to the positive cone generated by $\tilde{\tau}^+$ and \hat{v}^+ ;
- the vector field $\lambda^+|_{\text{Int } U^-}$ belongs to the positive cone generated by $\tilde{\tau}^+$ and $-\hat{v}^+$.

Let us pick a metric on P for which the vector fields $\tilde{\tau}^+$ and \hat{v}^+ are orthogonal and the vector fields $\tilde{\tau}^+$ and $\tilde{\lambda}^+$ have length 1. By rescaling, if necessary, the vector field \hat{v}^+ we can arrange that it has length 1 as well. Let us denote by $\theta(v)$ the angle between the vectors τ^+ and λ^+ in this metric. If we construct

the virtual vertical bundle ${}^\Phi\text{Vert}$ with this choice of the metric and the angle function θ , then the condition $\Delta_\Phi(\tilde{\lambda}) = \tilde{\lambda}$ will be satisfied. \square

In all our results below concerning an extension of a holonomic FLIF from a neighborhood of a closed set A we will always assume that over $\mathcal{O}p A$ all the necessary special choices are made to ensure the conclusion of Lemma 3.3.1: the virtual vertical bundle ${}^\Phi\text{Vert}$ coincides with Vert and the isomorphism $\Delta_\Phi : \text{Vert} \rightarrow {}^\Phi\text{Vert} = \text{Vert}$ is the identity, and hence, according to Lemma 3.2.2, we have $\Gamma_\Phi = \Pi_{\widehat{\Phi}}$, where $\widehat{\Phi} = \Phi^{\text{Norm}}$.

3.4 Balanced and well balanced FLIFs

We call a FLIF Φ *balanced* if the compositions

$$\text{Norm} \xrightarrow{\Gamma_\Phi} \text{Vert} \hookrightarrow TW|_V \quad \text{and} \quad \text{Norm} \xrightarrow{\Pi_{\widehat{\Phi}}} {}^\Phi\text{Vert} \hookrightarrow TW|_V$$

are homotopic in the space of injective homomorphisms $\text{Vert} \rightarrow TW|_V$. Here we denote by $\widehat{\Phi}$ the FLIF Φ^{Norm} . If Φ is holonomic over $\mathcal{O}p A \subset W$ then we say that Φ is *balanced relative A* if the homotopy can be made fixed over A .

Lemma 3.2.2 shows that the balancing condition is equivalent to the requirement that the composition $\text{Vert} \xrightarrow{\Delta_\Phi} {}^\Phi\text{Vert} \hookrightarrow TW|_V$ is homotopic to the inclusion $\text{Vert} \hookrightarrow TW|_V$ in the space of injective homomorphisms $\text{Vert} \rightarrow TW|_V$.

Lemma 3.1.2 shows that a holonomic Φ is balanced. Moreover, it is balanced relative to any closed subset $A \subset W$.

We say that a FLIF Φ is *well balanced* if it is prepared and the isomorphisms $\Pi_\Phi, \Gamma_\Phi : \text{Norm} \rightarrow \text{Vert}$ are homotopic as isomorphisms. Similarly we define the notion of a FLIF well balanced relative to a closed subset A .

It is not immediately clear from the definition that a well balanced FLIF is balanced. The next lemma shows that this is still the case.

3.4.1. A well balanced FLIF is balanced.

Proof. We need to check that over $V \setminus \Sigma$ we have $\Pi_\Phi = \Phi^2 \circ K^{-1}$ and $\Pi_{\widehat{\Phi}} = \Phi^2 \circ \widehat{K}^{-1}$, where K is the projection $\text{Norm} \rightarrow \text{Vert}$ and \widehat{K} is the projection $\text{Norm} \rightarrow {}^\Phi\text{Vert}$. We have $\widehat{K} = T \circ K$, where $T : \text{Vert} \rightarrow {}^\Phi\text{Vert}$ is the

projection along TV . Hence, we have $\Pi_{\widehat{\Phi}} = \Pi_{\Phi} \circ T$ which implies, in particular, that the projection T is non-degenerate over the whole V . Hence, the composition of the projection operator T with the inclusion ${}^{\Phi}\text{Vert} \xrightarrow{i} TW|_V$ is homotopic to the inclusion $\text{Vert} \xrightarrow{j} TW|_V$ as injective homomorphisms, and so do the compositions $i \circ \Pi_{\widehat{\Phi}}$ and $j \circ \Pi_{\Phi}$. \square

Note that for the codimension 1 case, i.e. when $n = 1$ the well balanced condition for a prepared FLIF is very simple:

3.4.2. (Well-balancing criterion in codimension 1) *Suppose $\dim \zeta = 1$. Then any prepared ζ -FLIF Φ is well balanced if and only if at one point $v \in V \setminus \Sigma$ of every connected component of V the map*

$$(\Pi_{\Phi})_v \circ (\Gamma_{\Phi})_v^{-1} : \text{Vert}_v \rightarrow \text{Vert}_v$$

is a multiplication by a positive number. The same statement holds also in the relative case.

3.4.3. (Well balanced FLIFs and folded isotopy) *Let Φ be a well balanced FLIF. Let $h_s : W \rightarrow W$ be a diffeotopy, ζ_s a family of n -dimensional distributions on W , and $\Theta_s : \zeta_0 \rightarrow \zeta_s$ a family of bundle isomorphisms covering h_s , $s \in [0, 1]$, such that $h_0 = \text{Id}$ and for each $s \in [0, 1]$*

- *submanifold $V_s := h_s(V) \subset W$ is folded with respect to ζ_s along $\Sigma_s := h_s(\Sigma)$;*
- *$dh_s(\zeta_0 \cap TV) = dh_s(\zeta_s) \cap TV_s$;*
- *$dh_s|_{\zeta_0 \cap TV} = \Theta_s|_{\zeta_0 \cap TV}$.*

Then the push-forward ζ_s -FLIF $\Phi_s := (h_s, \Theta_s)_ \Phi$, $s \in [0, 1]$, is well balanced.*

Proof. By assumption $V(\Phi_s)$ is folded with respect to ζ_s . Next, we observe that all co-orientations cannot change in the process of a continuous deformation, and similarly, the isomorphisms Π_{Φ_s} and Γ_{Φ_s} vary continuously, and hence remain homotopic as bundle isomorphisms $\text{Norm}(\Phi_s) \rightarrow \text{Vert}(\Phi_s)$. Thus the well balancing condition is preserved. \square

Note that if Φ is balanced then the homomorphism $\Delta_{\Phi} : \zeta|_V \rightarrow {}^{\Phi}\text{Vert}$ composed with the inclusion ${}^{\Phi}\text{Vert} \hookrightarrow TW$ extends to an injective homomorphism $F : \zeta \rightarrow TW$. Then $(\text{Id}, F)_* \Phi$ is a ν -FLIF extending the local ν -FLIF $\widehat{\Phi}$. Here we denoted by $\nu := F(\zeta)$.

3.4.4. *The ν -FLIF $\widehat{\Phi} = \Phi^{\text{Norm}}$ on $\mathcal{O}p V$ is well balanced.*

Proof. We already proved in 3.2.1 that $\widehat{\Phi}$ is prepared. Let us show that $\Pi_{\widehat{\Phi}} = \Gamma_{\widehat{\Phi}}$. According to the definition of the push-forward operator we have $\Gamma_{\widehat{\Phi}} = \Delta_{\Phi} \circ \Gamma_{\Phi}$. But according to Lemma 3.2.2 we have $\Delta_{\Phi} \circ \Gamma_{\Phi} = \Pi_{\widehat{\Phi}}$. \square

Consider a ζ -FLIF Φ . Suppose there exists a $(k+1)$ -dimensional submanifold $Y \subset W$, $Y \supset V$, such that

- Y is transverse to ζ ;
- the line field $\mu|_V \subset \text{Vert}$ is an eigenspace field for Φ^2 , where we denoted $\mu := \zeta \cap TY$;
- $\Phi^2|_{N:=\mu^\perp|_V}$ is non-degenerate, where μ^\perp is the orthogonal complement to μ in $\zeta|_Y$.

Consider the restriction μ -FLIF $\widetilde{\Phi} = \Phi|_Y$ defined as follows: $\widetilde{\Phi}^0 = \Phi^0|_Y$, $\widetilde{\Phi}^1$ is the projection of Φ^1 along μ^\perp , $\widetilde{\Phi}^2 = \Phi^2|_\mu$, $\widetilde{\lambda} = \lambda$. Note that we have $V(\widetilde{\Phi}) = V$ and $\Sigma(\widetilde{\Phi}) = \Sigma$.

We will assume that the bundle N is orthogonal to TY . Under this assumption we have $\Gamma_{\Phi}(N) = N$. The next criterion for a FLIF to be well-balanced is immediate from the definition.

3.4.5. *If $\widetilde{\Phi}$ is prepared then so is Φ . If $\widetilde{\Phi}$ is well balanced and $\Phi^2|_N = \Gamma_{\Phi}|_N$ then Φ is well balanced as well.*

3.5 Pleating a FLIF

We adjust in this section the pleating construction defined in Section 2.2 for submanifolds to make it applicable for framed well balanced FLIFs. Let Φ be a well balanced ζ -FLIF. We will use here the following notation from Section 2.2:

- $S \subset V_i \subset V \setminus \Sigma, i = 0, \dots, n$, is a closed cooriented codimension 1 submanifold;
- $U = S \times [-\delta, \delta] \supset S = S \times 0$ is a tubular δ -neighborhood of S in V_i ;
- $\nu^+ \in \zeta$ is a unit vector field defined over a neighborhood Ω of U in W ;

- $g : S \times [-\delta, \delta] \times [-\epsilon, \epsilon] \rightarrow \Omega \hookrightarrow W$ is an embedding such that $\frac{\partial g}{\partial u}(s, t, u) = \nu^+(g(s, t, u))$, $(s, t, u) \in S \times [-\delta, \delta] \times [-\epsilon, \epsilon]$, which maps $S \times 0 \times 0$ onto S and $S \times [-\delta, \delta] \times 0$ onto U ;
- $\Gamma \subset P := [-1, 1] \times [-1, 1]$ is an embedded connected curve which near ∂P coincides with the line $\{u = 0\}$;
- $\tilde{V} \subset W$ is the result of Γ -pleating of V over S in the direction of the vector field ν_+ .

We will make the following additional assumptions:

- * the splitting $\text{Vert}|_S = \text{Vert}_+|_S \oplus \text{Vert}_-|_S$ is extended to a splitting $\zeta = \zeta_+ \oplus \zeta_-$ over the neighborhood $\Omega \subset W$;
- * the vector field ν^+ is a section of either $\zeta_-|_\Omega$ or $\zeta_+|_\Omega$;
- * the vector field $\nu_+|_U$ is an eigenvector field for Φ^2 ;
- * $\text{Norm}(\Phi)|_U = \text{Vert}(\Phi)|_U$ and $\Delta_\Phi|_{\text{Vert}|_U} = \text{Id}$.

There exists a diffeotopy $h_s : W \rightarrow W$ supported in Ω connecting Id with a diffeomorphism h such that $h(V) = \tilde{V}$. We denote $\tilde{U} = \tilde{U}_\Gamma := h_1(\Gamma)$. Let $\Psi_s : \zeta \rightarrow \zeta$, $s \in [0, 1]$, be a family of isomorphisms covering h_s which preserve Vert_\pm and ν^+ .

The manifold \tilde{U} is folded with respect to ζ with the fold $\tilde{S} = \bigcup_1^{2N} \tilde{S}_j$ where $\tilde{S}_j = h_1(S_j)$, where $S_j = S \times t_j$, $-\delta < t_1 < \dots < t_{2N} < \delta$. Over \tilde{S} we have $\tilde{\tau} = \nu = T\tilde{V} \cap \zeta$.

Consider the push-forward FLIF $\overline{\Phi} := (h_1, \Psi_1)_* \Phi$. Though the manifold $V(\overline{\Phi}) = \tilde{V}$ is folded with respect to ζ , it is not prepared. We will modify $\overline{\Phi}$ to a prepared FLIF $\tilde{\Phi} = \text{Pleat}_{S, \nu^+, \Gamma}(\Phi)$ as follows.

Let $\tilde{c} : \tilde{U} \rightarrow \mathbb{R}$ be a function which on $\partial\tilde{U} = \partial U$ coincides with the eigenvalue function of the operator Φ^2 for the eigenvector field ν^+ , and have the fold $\tilde{S} := \bigcup_1^{2N} \tilde{S}_j$ as its regular 0-level. We call component of $\tilde{U} \setminus \tilde{S}$ *positive* or *negative* depending on the sign of the function \tilde{c} on this component. We then define

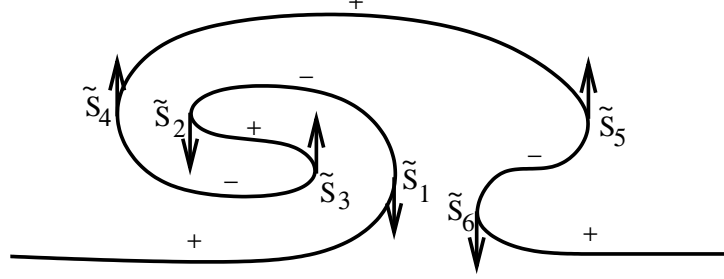


Figure 11: Γ -pleating of a well balanced FLIF

- $\tilde{\Phi}^1 = \overline{\Phi}^1$;
- $\tilde{\Phi}^2|_{\nu^\perp} = \overline{\Phi}^2|_{\nu^\perp}$;
- $\tilde{\Phi}^2(\nu^+) = \tilde{c}\nu^+$;
- $\lambda^+(\tilde{\Phi}^2) = \pm\nu^+$, where the sign is chosen in such way that the vector field $\lambda^+(\tilde{\Phi}^2)$ define an inward coorientation of positive components of $\tilde{U} \setminus \tilde{S}$, see Fig. 11.

We say that $\tilde{\Phi} = \text{Pleat}_{S,\nu^+,\Gamma}(\Phi)$ is obtained from Φ by Γ -pleating over S in the direction of the vector field ν^+ see Fig. 11.

3.5.1. *The FLIF $\tilde{\Phi}$ is well balanced.*

Proof. Consider the $(k+1)$ -dimensional manifold

$$Y := g(S \times [-\delta, \delta] \times [-\epsilon, \epsilon]) \subset W.$$

Then Y is transverse ζ and $\zeta \cap TY = \nu$. We also note that the orthogonal complement ν^\perp of $\nu \in \zeta$ is orthogonal to TY , $\tilde{\Phi}^2|_{\nu^\perp} = \Pi_{\tilde{\Phi}}|_{\nu^\perp} = \Gamma_{\tilde{\Phi}}|_{\nu^\perp}$. According to 3.4.5 it is sufficient to check that the restriction $\hat{\Phi} := \tilde{\Phi}|_Y$ is well balanced, rel. ∂Y . First, we need to check that this restriction is prepared. By construction, $\tilde{V} = V(\hat{\Phi})$ is folded with respect to ν and the vector field $\lambda^+(\hat{\Phi}) = \lambda^+(\tilde{\Phi})$ defines the index coorientation of \tilde{S} in \tilde{V} . Next, we need to check that the vector field $n^+(\hat{\Phi}) = n^+(\tilde{\Phi}) = \Gamma_{\tilde{\Phi}}^{-1}(\lambda^+(\tilde{\Phi}))$ defines the characteristic coorientation of the fold. It is sufficient to consider the case when S is the point, and hence $\dim Y = 2$. The general picture is then

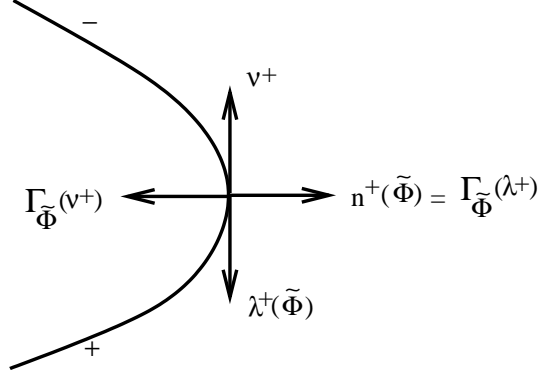


Figure 12: Vector $n^+(\tilde{\Phi})$ determines the characteristic coorientation of the fold

obtained by taking a direct product with S . Note that the characteristic coorientation of the fold \tilde{S}_j is given by the vector field $\frac{\partial}{\partial t}$ if j is odd, and by $-\frac{\partial}{\partial t}$ if j is even. Consider first the case when j is odd, see Fig. 12. Then if the lower branch of the parabola is positive then the vector field $\Gamma_{\tilde{\Phi}}(\nu^+)$ defines the same coorientation as the vector field $-\frac{\partial}{\partial t}$. But in this case $\lambda^+ = -\nu^+$, and hence $\Gamma_{\tilde{\Phi}}(\lambda^+)$ defines the characteristic coorientation of the fold. The other cases can be considered in a similar way. Finally, we use Lemma 3.4.2 to conclude that $\tilde{\Phi}$ is well balanced relative the boundary ∂Y . \square

In order to extend the Γ -pleating operation to framed well-balanced FLIFs we need to impose additional constraints on the choice of the vector field ν^+ and the curve Γ , see Fig. 13. For each $j = 1, \dots, 2N$ denote by σ_j the proportionality coefficient in $\lambda^+|_{\tilde{S}_j} = \sigma_j \nu^+|_{\tilde{S}_j}$, $\sigma_j = \pm 1$. Then we require that

- (α) if \tilde{S}_j and \tilde{S}_{j+1} , $j = 1, \dots, 2N - 1$ bound a negative component of $\tilde{U} \setminus \tilde{S}$ then $\sigma_j = \sigma_{j+1}$;
- (β) if the component bounded by \tilde{S}_1 and \tilde{S}_2 is positive then $\sigma_1 = \sigma_{2N} = \pm 1$ for $\nu^+ = \pm \xi^i$.

3.5.2. (Pleating a framed FLIF) *If ν^+ and Γ satisfy the above conditions, then given a framed well balanced FLIF (Φ, ξ) the FLIF $\tilde{\Phi} = \text{Pleat}_{S, \nu^+, \Gamma}$ admits a framing $\tilde{\xi}$, where the framing $\tilde{\xi}$ coincides with ξ outside \tilde{U} .*

Proof. The proof is illustrated on Fig. 13. In the case $\nu^+ \in \text{Vert}_+$ the pleating construction adds a 1-dimensional negative eigenspace to Vert_- restricted

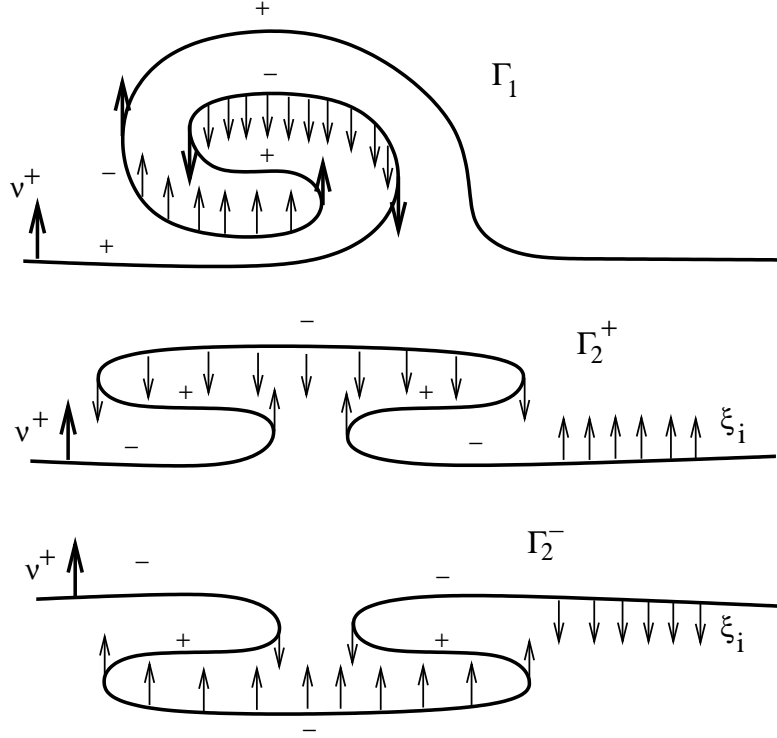


Figure 13: Framing of a Γ -pleated FLIF

to negative components $\tilde{U} \setminus \tilde{S}$. Condition (α) then allows us to frame this 1-dimensional space either with $\xi^{i+1} := \sigma_j \nu^+$. Similarly, if $\nu^+ \in \text{Vert}_-$ (or, equivalently when the component bounded by \tilde{S}_1 and \tilde{S}_2 is positive) then the pleating construction removes the negative eigenspace generated by ν^+ on positive components. The remaining negative components bounded \tilde{S}_{2j} and \tilde{S}_{2j+1} , $j = 1, \dots, N-1$, can be framed with $\tilde{\xi} := (\xi_1, \dots, \sigma_{2j} \xi_i)$. Condition (β) ensures that the existing framing in the complement of U satisfies the necessary boundary conditions on \tilde{S}_1 and \tilde{S}_{2N} . \square

3.5.3. *Given any framed well balanced FLIF (Φ, ξ) , one of the curves Γ_1, Γ_2^\pm shown on Fig. 6 can always be used as the curve Γ to produce a framed well balanced FLIF $(\tilde{\Phi}, \tilde{\xi})$ by a Γ -pleating.*

Proof. Indeed, as it follows from Criterion 3.5.2, the curve Γ_1 can always be used if $\nu^+|_S \in \text{Vert}_+$, while if $\nu^+|_S \in \text{Vert}_-$ then the curve Γ_2^\pm can be used in the case $\nu^+ = \pm \xi^i$, see Fig. 13. \square

The next proposition is a corollary of Theorem 2.2.1 and the results discussed in the current section.

3.5.4. (Pleated isotopy of framed well balanced FLIFs) *Let ζ_s , $s \in [0, 1]$, be a family of n -dimensional distributions on W , and (Φ, ξ) a framed well-balanced ζ_0 -FLIF with $V(\Phi) = V \subset W$. Then there exist*

- *a framed well balanced ζ_0 -FLIF $\tilde{\Phi}$ obtained from Φ by a sequence of pleatings, and*
- *a C^0 -small isotopy $h_s : V \rightarrow W$, $s \in [0, 1]$ such that h_0 is the inclusion $V \hookrightarrow W$ and $\tilde{V}_s := h_s(V(\tilde{\Phi}))$ is folded with respect to ζ_s along $\tilde{\Sigma}_s := h_s(\Sigma(\tilde{\Phi}))$.*

If Φ is holonomic over $\mathcal{O}p A$ then one can arrange that $\tilde{\Phi} = \Phi$ on $\mathcal{O}p A$ and that the homotopy h_s is fixed over $\mathcal{O}p A$.

Proof. According to Theorem 2.2.1 there exists a manifold \tilde{V} for which the isotopy with the required properties does exist. This manifold can be constructed beginning from V by a sequence of Γ_0^+ -pleatings along the boundaries of balls embedded into $V \setminus \Sigma$, in the direction of vector fields which extend to these balls. The latter property allows us to deform these vector fields into vector fields contained in Vert_+ or Vert_- (we need to use Vert_- only if $\dim \text{Vert}_+ = 0$). Moreover, when using $\nu^+ \in \text{Vert}_-|_{V_i}$ and when $i = \dim \text{Vert}_-|_{V_i} > 1$ we can deform it further into the last vector ξ^i of the framing. In the case $i = 1$ we can deform ν^+ into $\pm \xi^i$, but we cannot, in general, control the sign. Note that we need to use this case only if $n = 1$. As it was explained in Remark 2.2.2, we can replace at our choice each Γ_0^+ -pleating in the statement of Theorem 2.2.1 by any of the Γ -pleatings with $\Gamma = \Gamma_1, \Gamma_2^\pm$. But according to Lemma 3.5.3 one can always use one of these curves to pleat in the class of framed well balanced FLIFs. It remains to observe that if Φ is holonomic over $\mathcal{O}p A$ then all the constructions which we used in the proof can be made relative to $\mathcal{O}p A$. \square

3.6 Stabilization

Let Φ be a ζ -FLIF. Suppose that we are given a connected domain $U \subset V \setminus \Sigma$ with smooth boundary such that the bundles $\text{Vert}_\pm|_U$ are trivial. Let C be an exterior collar of $\partial U \subset V \setminus \Sigma$. We set $U' := U \cup C$.

Let us assume that U is contained in V^i . If $i < n$ we choose a section θ^+ of the bundle Vert_+ over U' and we define a *negative stabilization* of Φ over U as a FLIF $\tilde{\Phi} = \text{Stab}_{U, \theta^+}^-(\Phi)$ such that

- $\tilde{\Phi}^1 = \Phi^1$;
- $\tilde{\Phi}^2 = \Phi^2$ over $V \setminus U'$;
- $\Sigma(\tilde{\Phi}) = \Sigma(\Phi) \cup \partial U$; $\text{Int } U \subset V^{i+1}(\tilde{\Psi})$;
- $\text{Vert}_-(\tilde{\Phi})|_{\text{Int } U} = \text{Span}(\text{Vert}_-(\Phi)|_{\text{Int } U}, \theta^+)$;
- $\lambda^+(\tilde{\Phi})|_{\partial U} = \theta^+$.

We will omit a reference to θ in the notation and write simply $\text{Stab}_U^-(\Phi)$ when this choice will be irrelevant.

Note that in order to construct $\tilde{\Phi}^2$ on U' which ensures these property we need to adjust the background metric on ζ to make θ^+ an eigenvector field for Φ^2 corresponding the eigenvalue $+1$. The vector field θ^+ remains the eigenvector field for $\tilde{\Phi}^2$ but the eigenvalue function is changed to $c : U' \rightarrow [-1, 1]$, where c is negative on U , equal to 1 near $\partial U'$ and has ∂U as its regular 0-level.

If the FLIF Φ is framed by $\xi = (\xi^1, \dots, \xi^i)$ then $\tilde{\Phi}$ can be canonically framed by $\tilde{\xi}$ such that $\tilde{\xi} = \xi$ over $V \setminus U$ and $\text{Vert}_-(\tilde{\Phi})|_{\text{Int } U}$ is framed by $\tilde{\xi} := (\xi^1, \dots, \xi^i, \theta^+)$ and we define

$$\text{Stab}_U^-(\Phi, \xi) := (\text{Stab}_{U, \xi^i}^-(\Phi), \tilde{\xi}),$$

In the case when $U \subset V_i$ and $i > 0$ we can similarly define a *positive stabilization* of Φ over U as a FLIF $\tilde{\Phi} = \text{Stab}_{U, \theta}^+(\Phi)$, where θ is a section of Vert_- over U' such that

- $\tilde{\Phi}^1 = \Phi^1$;
- $\tilde{\Phi}^2 = \Phi^2$ over $V \setminus U'$;
- $\Sigma(\tilde{\Phi}) = \Sigma(\Phi) \cup \partial U$; $\text{Int } U \subset V^{i-1}(\tilde{\Psi})$;
- $\text{Vert}_+(\tilde{\Phi})|_{\text{Int } U} = \text{Span}(\text{Vert}_+(\Phi)|_{\text{Int } U}, \theta)$;
- $\lambda^+(\tilde{\Phi})|_{\partial U} = \theta$.

If Φ is framed by a framing $\xi = (\xi^1, \dots, \xi^i)$ then we will always choose $\theta^+ = \xi^i|_U$ and define a positive stabilization by the formula

$$\text{Stab}_U^+(\Phi, \xi) := (\text{Stab}_{U, \xi^i}^+(\Phi), \tilde{\xi}),$$

where $\tilde{\xi}|_{\text{Int } U} = (\xi^1, \dots, \xi^{i-1})$.

3.6.1. (Balancing via stabilization) *Any FLIF can be stabilized to a balanced one. If Φ is balanced and $\chi(U) = 0$ then $\text{Stab}_U^\pm(\Phi)$ is balanced as well. The statement holds also in the relative form.*

Proof. The obstruction for existence of a fixed over $A \subset V$ homotopy between two monomorphisms $\Psi_1, \Psi_2 : \text{Norm} \rightarrow TW|_V$ is an n -dimensional cohomology class $\delta(\Psi_1, \Psi_2; V, A) \in H^k(V, A; \pi_k(V_n(\mathbb{R}^{n+k})))$, or more precisely a cohomology class with coefficients in the local system $\pi_k(V_n(T_v W))$, $v \in V$. Note that $\pi_k(V_n(\mathbb{R}^{n+k})) = \mathbb{Z}$ if k is even or $n = 1$ and $\mathbb{Z}/2$ otherwise. It is straightforward to see that

$$\delta(\Delta(\Phi), \Delta(\text{Stab}_U^\pm(\Phi); U, \partial U) = \begin{cases} \chi(U)\Theta, & k \text{ is even;} \\ \pm\chi(U)\Theta, & k \text{ is odd,} \end{cases}$$

for an appropriate choice of a generator Θ of $H^k(U, \partial U; \pi_k(V_n(\mathbb{R}^{n+k})))$. Hence, stabilization over a domain with vanishing Euler characteristic does not change the obstruction class $\delta(\Gamma(\Phi), \Delta(\Phi))$ and with the exception of the case $k = n = 1$ this obstruction class can be changed in an arbitrary way by an appropriate choice of U . Indeed, if $k > 1$ then one can take as U either the union of l copies of n -balls or a regular neighborhood of an embedded bouquet of l circles (comp. a similar argument in [EGM11]). If $k = 1$ and $n > 1$ then the sign issue is irrelevant because the obstruction is $\mathbb{Z}/2$ -valued. If $k = n = 1$ then one may need two successive stabilizations in order to balance a FLIF. Indeed, the domain U in this case is a union of some number l of intervals, and hence $\chi(U) = l$. Thus the positive stabilization increases the obstruction class by l , while the negative one decreases it by l . Suppose, for determinacy, we want to stabilize over a domain in V_0 . If we need to change the obstruction class by $-l$ then we just negatively stabilize over the union of l intervals. If we need to change it by $+l$ we first negatively stabilize over one interval I and then positively stabilize over the union of $l + 1$ disjoint intervals in I . \square

3.7 From balanced to well balanced FLIFs

3.7.1. (From balanced to well balanced) *Let (Φ, ξ) be a balanced framed ζ -FLIF which is holonomic over a neighborhood of a closed subset $A \subset W$. Then there exists a framed well-balanced FLIF (Φ', ξ') which coincides with Φ over $\mathcal{O}p A$. In addition, $V(\Phi')$ is obtained from $V(\Phi)$ via a C^0 -small, fixed on $\mathcal{O}p A$ isotopy.*

Proof. There exists a family of monomorphisms $\Psi_s : \text{Vert} \rightarrow TW$, $s \in [0, 1]$, connecting $\text{Vert} \xrightarrow{\Delta_\Phi} \Phi \text{Vert} \hookrightarrow \tau$ and the inclusion $j : \text{Vert} \hookrightarrow \tau$. The homotopy can be chosen fixed over $\mathcal{O}p A$. The family Ψ_s can be extended to a family of monomorphisms $\zeta \rightarrow TW$. We will keep the notation Ψ_s for this extension. Denote $\zeta_s := \Psi_s(\zeta)$, $s \in [0, 1]$. Thus $\zeta_1 = \zeta$ and ζ_0 is an extension to W of the bundle Norm^Φ . Lemma 3.4.4 then guarantees that the push-forward ζ_0 -FLIF $(\text{Id}, \Psi_0)_*(\Phi, \xi)$ is well balanced. According to Theorem 3.5.4 there exists a well balanced framed ζ_0 -FLIF $(\widehat{\Phi}, \widehat{\xi})$ where $\widehat{V} = V(\widehat{\Phi})$ is obtained from V by a C^0 -small isotopy which is fixed outside a neighborhood of V and over a neighborhood of A , and a C^0 -small supported in $(\mathcal{O}p \widehat{V}) \setminus A$ isotopy g_s starting with $g_0 = \text{Id}$ such that for each $s \in [0, 1]$ the manifold $\widehat{V}_s := g_s(\widehat{V})$ is folded with respect to ζ_s along $\widehat{\Sigma}_s = g_s(\widehat{\Sigma})$. There exists a family of bundle isomorphisms $\Theta_s : \zeta_0 \rightarrow \zeta_s$ covering the diffeotopy h_s and such that $\Theta_0 = \text{Id}$ and $\Theta_s = dg_s$ over the line bundle $TV|_{\widehat{\Sigma}} \cap \zeta_0$. The homotopy Θ_s can be chosen fixed over $\mathcal{O}p A$. Then, according to Lemma 3.4.3, the push-forward ζ -FLIF $(g_1, \Theta_1)_*(\widehat{\Phi}, \widehat{\xi})$ is well balanced relative A . \square

3.8 Formal extension

3.8.1. (Formal extension theorem) *Any framed ζ -FLIF (Φ, ξ) on $\mathcal{O}p A \subset W$ extends to a framed ζ -FLIF $(\widetilde{\Phi}, \widetilde{\xi})$ on the whole manifold W .*

The proof is essentially Igusa's argument in [Ig87] (see pp.438-442).

We begin with the following lemma which will be used as an induction step in the proof.

3.8.2. (Decreasing the negative index) *Let $j = 1, \dots, n$. Suppose W is a cobordism between $\partial_- W$ and $\partial_+ W$, and for a framed FLIF (Φ, ξ) on W one has $V^i = \emptyset$ for $i > j$. Then there exists a framed FLIF $(\widetilde{\Phi}, \widetilde{\xi})$ such that*

- $\Phi = \tilde{\Phi}$ on $\mathcal{O}_p(\partial_- W)$;
- $V^i(\tilde{\Phi}) \cap \partial_+ W = \emptyset$ for $i \geq j$.

Proof of 3.8.2. To prove the claim we recall that the j -dimensional bundle Vert_- over V^j is trivialized by the framing $\xi = (\xi^1, \dots, \xi^j)$, and $\xi^j|_{\Sigma^{j-1}} = \lambda^+$. We can extend the vector field ξ^j to a neighborhood G of V^j in $V^{j-1} \cup V^j \cup \Sigma^{j-1}$ as a unit vector field in V_+^{j-1} . Let X^j be the self-adjoint linear operator $\text{Vert}_+ \rightarrow \text{Vert}_+$ defined on the neighborhood G which orthogonally projects Vert_+ to the line bundle spanned by ξ^j . Choose neighborhoods $H_- \supset \partial_- W$ and $H_+ \supset \partial_+ W$ in W with disjoint closures and consider a cut-off function $\theta : V \rightarrow \mathbb{R}_+$ which is equal to 0 on $(V \cap H_-) \cup (V \setminus G)$ and equal to 1 on $V^j \cap H_+$. Set $\tilde{\Phi}^2 := \Phi^2 + C\theta X^j$. Then for a sufficiently large $C > 0$ the self-adjoint operator $\tilde{\Phi}^2$ coincides with Φ^2 on $V \cap H_-$, has negative index $\leq j$ everywhere, and $< j$ on $V^j \cap H_+$, see Fig.14.

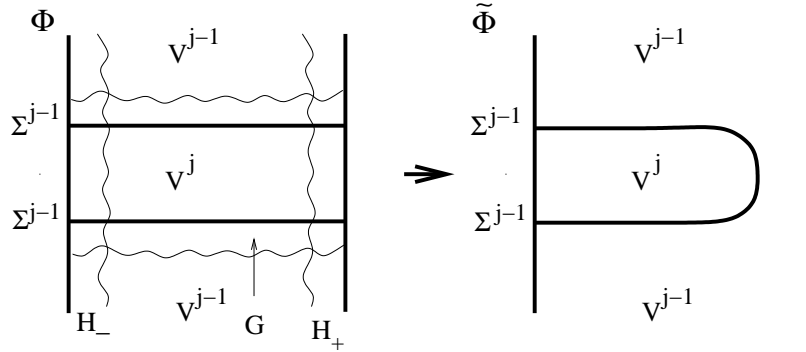


Figure 14: Decreasing the negative index

The kernel of $\tilde{\Phi}^2$ on $\Sigma^{j-1}(\tilde{\Phi})$ is generated by ξ^j , and hence there is a canonical way to define the vector field $\lambda^+(\tilde{\Phi})|_{\Sigma^{j-1}}$. Note that $\tilde{\Phi} = \Phi$ in the complement $V \setminus V^{j-1}(\Phi) \cup V^j(\Phi) \cup \Sigma^{j-1}(\Phi)$ and at each point $v \in V^{j-1}(\Phi) \cup V^j(\Phi) \cup \Sigma^{j-1}(\Phi)$ the negative eigenspace $\text{Vert}_-(\tilde{\Phi})$ coincides either with $\text{Vert}_-(\Phi)$, or with the span of the vectors ξ^1, \dots, ξ^{j-1} . Hence, the framing ξ of Φ determines a framing $\tilde{\xi}$ of $\tilde{\Phi}$. \square

Proof of 3.8.1. Let $\Phi = (\Phi^0, \Phi^1, \Phi^2, \lambda^+)$. Without loss of generality we can assume that (Φ, ξ) is defined on an $(n + k)$ -dimensional domain $C \subset W$, $\text{Int } C \supset A$, with smooth boundary. Note that if $\Phi^2|_{V \cap \partial C}$ is positive definite, then the extension obviously exist. Indeed, we can extend Φ^1 in

any generic way to W , and then extend Φ^2 as a positive definite operator on Vert . We will inductively reduce the situation to this case. Let $C' \subset \text{Int } C$ be a smaller domain such that $A \subset \text{Int } C'$. Let us apply 3.8.2 to the cobordism $W_{(0)} = C \setminus \text{Int } C'$ between $\partial_- W_{(0)} = \partial C'$ and $\partial_+ W_{(0)} = \partial C$ and to the restriction $(\Phi, \xi)|_{W_{(0)}}$ in order to modify $(\Phi, \xi)|_{W_{(0)}}$ into a framed FLIF $(\Phi_{(0)}, \xi_{(0)})$ which coincides with (Φ, ξ) near $\partial_- W_{(0)}$ and such that $V^n(\Phi_{(0)}) \cap (\partial_+ W_{(0)}) = \emptyset$. Then for a sufficiently small tubular neighborhood $W_{(1)}$ of $\partial_+ W_{(0)}$ in $W_{(0)}$ we have $W_{(1)} \cap A = \emptyset$ and $W_{(1)} \cap V^n(\Phi_{(0)}) = \emptyset$. We view $W_{(1)}$ as a cobordism between $\partial_- W_{(1)} = \partial W_{(1)} \setminus \partial_+ W_{(0)}$ and $\partial_+ W_{(1)} = \partial_+ W_{(0)}$. Now we again apply 3.8.2 to the cobordism $W_{(1)}$ and $\Phi_{(0)}|_{W_{(1)}}$ and construct a framed FLIF $(\Phi_{(1)}, \xi_{(1)})$ on $W_{(1)}$ which coincides with $(\Phi_{(0)}, \xi_{(0)})$ near $\partial_- W_{(1)}$ and such that $V^i(\Phi_{(1)}) \cap \partial_+ W_{(1)} = \emptyset$ for $i \geq n - 1$. Continuing this process we construct a sequence of nested cobordisms $C \supset W_{(0)} \supset W_{(1)} \supset \cdots \supset W_{(n-1)}$ and a sequence of framed FLIFs $(\Phi_{(j)}, \xi_{(j)})$ on $W_{(j)}$, $j = 0, \dots, n - 1$, such that for all $j = 0, \dots, n - 1$

- $\partial_+ W_{(j)} = \partial C$;
- $(\Phi_{(j+1)}, \xi_{(j+1)})$ coincides with $(\Phi_{(j)}, \xi_{(j)})$ on $\mathcal{O}p(\partial_- W_{(j+1)})$;
- $V^i(\Phi_{(j)}) \cap \partial_+ W_{(j)} = \emptyset$ for $i \geq n - j$.

Let us also set $W_{(n)} = \emptyset$. Hence we can define a framed formal Igusa function $(\tilde{\Phi}, \tilde{\xi})$ over C by setting $(\tilde{\Phi}, \tilde{\xi}) = (\Phi, \xi)$ on C' and $(\tilde{\Phi}, \tilde{\xi}) = (\Phi_{(j)}, \xi_{(j)})$ on $W_{(j)} \setminus W_{(j+1)}$ for $j = 0, \dots, n - 1$. Note that the quadratic part $\tilde{\Phi}^2$ of $\tilde{\Phi}$ is positive definite on ∂C , and hence the framed formal Igusa function $\tilde{\Phi}$ can be extended to the whole W . \square

3.9 Integration near V

3.9.1. (Local integration of a well balanced FLIF) *Any well balanced \mathcal{F} -FLIF Φ can be made holonomic near V after a small perturbation near V . Namely, there exists a homotopy of well balanced FLIFs Φ_s , $s \in [0, 1]$, $s \in [0, 1]$, beginning with $\Phi_0 = \Phi$ with the following properties:*

- $V(\Phi_s) = V(\Phi)$, $\Sigma(\Phi_s) = \Sigma(\Phi)$ for all $s \in [0, 1]$;
- (Φ_s^2, λ_s^+) is C^0 -close to (Φ^2, λ^+) for all $s \in [0, 1]$;
- Φ_1 is holonomic on $\mathcal{O}p V$.

If for a closed subset $A \subset W$ the FLIF Φ is already holonomic over $\mathcal{O}p A \subset W$ then the homotopy can be chosen fixed over $\mathcal{O}p A$.

Proof. According to Lemma 2.1.1 there exist local coordinates (σ, t, z, y) in a neighborhood of Σ in W , where $\sigma \in \Sigma$, $x, z \in \mathbb{R}$ and $y \in \text{Ver}|_{\Sigma}$ such that the manifold V is given by the equations $z = x^2, y = 0$ and the foliation \mathcal{F} is given by the fibers of the projection $(\sigma, x, z, y) \rightarrow (\sigma, z)$. The vector field $\frac{\partial}{\partial x}$ generates the line bundle $\lambda = TV|_{\Sigma} \cap \text{Vert}$ and we can additionally arrange that $\frac{\partial}{\partial x}|_{\Sigma} = \lambda^+$. By a small C^0 -small perturbation of the operator Φ^2 (without changing it along Σ) we can arrange the the vector field $\frac{\partial}{\partial x}$ serves as an eigenvector field for Φ^2 in a neighborhood of Σ . We will keep the notation λ for the extended line field $\frac{\partial}{\partial x}$. Then the operator $\Phi^2 : \text{Vert} = \text{Ver} \oplus \lambda \rightarrow \text{Ver} \oplus \lambda$ can be written as $A \oplus c$, where A is a non-degenerate self-adjoint operator and c is an operator acting on the line bundle λ by multiplication by a function $c = c(\sigma, x)$ on $\mathcal{O}p \Sigma \subset V$ such that for all $\sigma \in \Sigma$ we have $c(\sigma, 0) = 0$, $d(\sigma) := \frac{\partial c}{\partial x}(\sigma, 0) > 0$. Define a function φ on $\mathcal{O}p \Sigma \supset W$ given by the formula

$$\varphi(\sigma, x, z, y) = \frac{d(\sigma)}{6}(x^3 - 3zx) + \frac{1}{2}\langle Ay, y \rangle. \quad (3)$$

Then $V(\varphi) = V \cap \mathcal{O}p \Sigma$ and the operator $d_{\mathcal{F}}^2 \varphi : \text{Ver} \oplus \lambda \rightarrow \text{Ver} \oplus \lambda$ is equal to $A \oplus \hat{c}$, where the operator \hat{c} acts on λ by multiplication by the function $d(\sigma)x$. Hence the operator functions $d^2 \varphi$ and Φ^2 coincides with the first jet along Σ , and therefore, one can adjust Φ^2 by a C^0 -small homotopy to make Φ^2 equal to $d^2 \varphi$ over $\mathcal{O}p \Sigma \subset W$. To extend φ to a neighborhood $\mathcal{O}p V \subset W$ we observe that the neighborhood of V in W is diffeomorphic to the neighborhood of the zero section in the total space of the bundle $\text{Vert}|_{V \setminus U}$. In the corresponding coordinates we define $\varphi(v, y) := \frac{1}{2}\langle \Phi^2(v)y, y \rangle$, $v \in V, y \in \text{Vert}_v$. On the boundary of the neighborhood of Σ where we already constructed another function, the two functions differ in terms of order $o(\|y\|^2)$. Hence they can be glued together without affecting $d_{\mathcal{F}}^2 \varphi$, and thus we get a leafwise Igusa function φ with $d_{\mathcal{F}}^2 \varphi = \Phi^2$. It remains to extend $\nabla_{\mathcal{F}} \varphi$ as a non-zero section of the bundle $T\mathcal{F}$ to the whole W . According to Lemmas 3.1.2 and 3.3.1 we have $\Gamma_{\varphi} = \Pi_{\varphi} = d_{\varphi}^2 = \Phi^2$. Then the well balancing condition for Φ implies that Γ_{φ} is homotopic (rel. $\mathcal{O}p A$) to Γ_{Φ} as isomorphisms $\text{Norm} \rightarrow \text{Vert}$. But this implies that there is a homotopy (rel. $\mathcal{O}p A$) of sections $\Phi_s^1 : W \rightarrow \text{Vert}, s \in [0, 1]$, connecting $\Phi_0^1 = \Phi^1$ and $\Phi_1^1 = \nabla_{\mathcal{F}} \varphi$ and such that the zero set remains regular and unchanged. \square

4 Proof of Extension Theorem 1.1.1

STEP 1. FORMAL EXTENSION. We begin with a leafwise framed Igusa function (φ_A, ξ_A) . Using 3.8.1 we extend it to a FLIF (Φ, ξ) on W .

All consequent steps are done without changing anything on $\mathcal{O}_p A$.

STEP 2. STABILIZATION. Using 3.6.1 we make (Φ, ξ) balanced.

STEP 3. FROM BALANCED TO WELL BALANCED. Using 3.7.1 we further improve (Φ, ξ) making it well balanced.

STEP 4. LOCAL INTEGRATION NEAR V . Using 3.9.1 we deform (Φ, ξ) without changing $V(\Phi)$ to make it holonomic near V .

STEP 5. HOLONOMIC EXTENSION TO W . Now on $W \setminus \mathcal{O}_p V$ we are in a position to apply Wrinkling Theorem 1.6B from [EM97] (see also [EM98], p.335) to extend the constructed $\varphi_{A \cup V}$ as a leafwise wrinkled map $\varphi : (W, \mathcal{F}) \rightarrow \mathbb{R}$. The wrinkles of φ of any index have the canonical framing and thus this completes the proof of Theorem 1.1.1.

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